On the γ-reflected processes with fBm input

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Abstract: Define a γ-reflected process $W_\gamma(t) = Y_H(t) - \gamma \inf_{s\in[0,t]} Y_H(s)$, $t \geq 0, \gamma \in [0,1]$ with $\{Y_H(t), t \geq 0\}$ a fractional Brownian motion with Hurst index $H \in (0,1)$ and a negative linear trend. In risk theory $R_\gamma(t) = u - W_\gamma(t), t \geq 0$ is the risk process with tax of a loss-carry-forward type and initial reserve $u \geq 0$, whereas in queueing theory W_1 is referred to as the queue length process. In this paper, we investigate the ruin probability and the ruin time of R_γ over a reserve dependent time interval.

Key Words: γ-reflected process; risk process with tax; ruin probability; ruin time; maximum losses; fractional Brownian motion; Piterbarg constant.

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1 Introduction

Let $\{X_H(t), t \geq 0\}$ be a standard fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$, i.e., X_H is a centered Gaussian process with almost surely continuous sample paths and covariance function

$$\mathrm{Cov}(X_H(t), X_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |\, t - s\,|^{2H}), \quad t, s \geq 0.$$

Define a γ-reflected process

$$W_\gamma(t) = Y_H(t) - \gamma \inf_{s\in[0,t]} Y_H(s), \quad t \geq 0, \tag{1}$$

where $\gamma \in [0,1]$ is the reflection parameter and $Y_H(t) = X_H(t) - ct, t \geq 0$ with some constant $c > 0$. In the actuarial literature $R_\gamma(t) = u - W_\gamma(t), t \geq 0, u \geq 0$ is referred to as the risk process with tax of a loss-carry-forward type; see, e.g., [2]. In queuing theory W_1 is referred to as the queue length process (or the workload process); see, e.g., [3, 17]. We refer to [4, 9, 10, 22, 23, 24, 12] for some recent studies of W_0. Next, define the *ruin time* of the γ-reflected process W_γ by

$$\tau_{\gamma,u} = \inf\{t \geq 0 : W_\gamma(t) > u\} \ \ (\text{with } \inf\{\emptyset\} = \infty). \tag{2}$$

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Further let $T_u, u \geq 0$ be a positive function and define the *ruin probability* over a reserve dependent time interval $[0, T_u]$ by

$$\psi_{\gamma,T_u}(u) := \mathbb{P}\left(\tau_{\gamma,u} \leq T_u\right).$$

Hereafter, $\psi_{\gamma,\infty}(u) = \mathbb{P}\left(\tau_{\gamma,u} < \infty\right)$ denotes the ruin probability over an infinite-time horizon.

The ruin time and the ruin probability for both $T_u = T \in (0, \infty)$ and $T_u = \infty$ for all u large are studied in [20, 19]; see also [10, 22, 23]. In [20] the exact asymptotics of $\psi_{\gamma,T}(u)$ and $\psi_{\gamma,\infty}(u)$ are derived, which combined with the results in [22] and [10] lead to the following interesting asymptotic equivalence

$$\psi_{\gamma,T}(u) = \mathcal{C}_{H,\gamma}\psi_{0,T}(u)(1 + o(1)), \quad u \to \infty \tag{3}$$

for any $T \in (0, \infty]$ and $\gamma \in (0, 1)$, with $\mathcal{C}_{H,\gamma}$ some known positive constant. The recent contribution [19] investigates the approximation of the conditional ruin time $\tau_{\gamma,u}|(\tau_{\gamma,u} < \infty)$. As shown therein the following convergence in distribution (denoted by $\xrightarrow{d}$)

$$\frac{\tau_{\gamma,u} - t_0 u}{A_{c,H} u^H}\Big|(\tau_{\gamma,u} < \infty) \xrightarrow{d} \mathcal{N} \tag{4}$$

holds as the initial reserve u tends to infinity for any $\gamma \in [0, 1)$, where $\mathcal{N}$ is an $N(0, 1)$ random variable and

$$t_0 = \frac{H}{c(1 - H)}, \quad A_{c,H} = t_0^{H+\frac{1}{2}} c^{-\frac{1}{2}}. \tag{5}$$

See also [23, 24, 25, 7] for related results. Of course, the ruin time and the ruin probability are studied extensively in the framework of other stochastic processes, see, e.g., [2, 15, 16, 14].

With motivation from [4] and [8], as a continuation of the aforementioned studies, in this contribution we shall analyze the ruin probability and the conditional ruin time of the γ-reflected process W_γ over the reserve dependent time interval $[0, T_u]$. Allowing the time horizon to be adjusted by the initial reserve level u of the portfolio is one of the new features in this contribution. The motivation for doing so is the insurance rational that if the company allocates a high initial reserve u to a specific insurance portfolio, then the time horizon that this portfolio is not ruined, say with at least 99% probability, should be closely related to the level u. Our investigation shows that considering a time horizon that depends on the initial reserve u leads to interesting theoretical results which are also of interest for future practical actuarial work.

As mentioned above, a novel aspect of this paper is that T_u will be a function changing with the initial reserve u according to three different scenarios defined with the help of (4). In Theorem 2.1 below we show that similar asymptotic equivalence as in (3) still holds for all the three scenarios. In Theorem 2.2 we derive a truncated Gaussian approximation for the (scaled) conditional ruin time over the long time horizon, whereas for the short and the intermediate time horizons an exponential approximation is possible.

We organize this contribution as follows: The main results are presented in Section 2 followed then by a section dedicated to the proofs.

2 Main Results

For the time horizon $[0, T_u]$ we shall consider three interesting scenarios which are specified by determining the relation between T_u and the initial reserve u. In view of (4) asymptotically (and roughly speaking) the mean of the ruin time equals $t_0 u$ and its standard deviation equals $A_{c,H} u^H$ (see also (5)). Therefore, for the choices of T_u, both $t_0 u$ and $A_{c,H} u^H$ should be utilized as scaling parameters, leading to the following three scenarios:

1) The short time horizon: $\lim_{u\to\infty} T_u/u = 0 =: s_0$;

2) The intermediate time horizon: $\lim_{u\to\infty} T_u/u = s_0 \in (0, t_0)$;

3) The long time horizon: $\lim_{u\to\infty} \frac{T_u - t_0 u}{A_{c,H} u^H} = x \in (-\infty, \infty]$.

Hereafter we shall write $\mathcal{N}$ for an $N(0,1)$ random variable with survival function $\Psi(\cdot)$, and

$$\mathcal{H}_\alpha = \lim_{S\to\infty} \frac{1}{S} \mathbb{E}\left(\exp\Bigl(\sup_{t\in[0,S]}\Bigl(\sqrt{2}X_{\alpha/2}(t) - t^\alpha\Bigr)\Bigr)\right) \in (0,\infty)$$

for the Pickands constant, where $X_{\alpha/2}$ is a standard fBm with Hurst index $\alpha/2 \in (0,1)$. Another important constant is Piterbarg's one defined by

$$\mathcal{P}_\alpha^b = \mathbb{E}\left(\exp\Bigl(\sup_{t\in[0,\infty)}\Bigl(\sqrt{2}X_{\alpha/2}(t) - (1+b)t^\alpha\Bigr)\Bigr)\right) \in (0,\infty),\ \alpha\in(0,2),\ b>0.$$

We refer to [26, 18, 6, 9, 5, 1, 11, 20, 8, 13, 27, 21, **?**] for properties and extensions of the Pickands and Piterbarg constants. As shown in [4] for the 0-reflected risk process W_0 with $H \in (0,1)$ we have:

(i) If $\lim_{u\to\infty} T_u/u = s_0 \in [0, t_0)$, then

$$\psi_{0,T_u}(u) = \mathcal{D}_H \left(\frac{u + cT_u}{T_u^H}\right)^{(\frac{1-2H}{H})_+} \Psi\left(\frac{u + cT_u}{T_u^H}\right)(1 + o(1)), \quad u \to \infty, \tag{6}$$

where

$$\mathcal{D}_H = \begin{cases} 2^{-\frac{1}{2H}}(H-c_0)^{-1}\mathcal{H}_{2H}, & \text{if } H<1/2, \\ \frac{2(1-c_0)}{(1-2c_0)}, & \text{if } H=1/2, \\ 1 & \text{if } H>1/2, \end{cases} \qquad \text{with } c_0 = \frac{cs_0}{1+cs_0}. \tag{7}$$

(ii) If $\lim_{u\to\infty} \frac{T_u - t_0 u}{A_{c,H}u^H} = x \in (-\infty, \infty]$, then

$$\psi_{0,T_u}(u) = \psi_{0,\infty}(u)\Phi(x)(1+o(1)), \quad u\to\infty, \tag{8}$$

where $\Phi(x) = 1-\Psi(x)$ and

$$\psi_{0,\infty}(u) = 2^{\frac{1}{2}-\frac{1}{2H}} \frac{\sqrt{\pi}}{\sqrt{H(1-H)}}\mathcal{H}_{2H}\left(\frac{c^H u^{1-H}}{H^H(1-H)^{1-H}}\right)^{1/H-1}\Psi\left(\frac{c^H u^{1-H}}{H^H(1-H)^{1-H}}\right)(1+o(1)). \tag{9}$$

Our first result below shows the asymptotic relation between the ruin probability ψ_{γ,T_u} of the γ-reflected process W_γ and that of the 0-reflected process W_0. Consequently, in the light of (i) and (ii) above the exact asymptotics as $u\to\infty$ of $\psi_{\gamma,T_u}(u)$ follows easily.

Theorem 2.1 *Let W_γ be the γ-reflected process defined in* (1) *with $H\in(0,1)$ and $\gamma\in(0,1)$. We have*
i) If $\lim_{u\to\infty} T_u/u = s_0 \in [0, t_0)$, then

$$\psi_{\gamma,T_u}(u) = \mathcal{M}_{H,\gamma}\psi_{0,T_u}(u)(1+o(1)), \quad u\to\infty, \tag{10}$$

where

$$\mathcal{M}_{H,\gamma} = \begin{cases} \mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}, & \textit{if } H<1/2, \\ \frac{2-2c_0}{2-2c_0-\gamma}, & \textit{if } H=1/2, \\ 1 & \textit{if } H>1/2. \end{cases}$$

ii) If $\lim_{u\to\infty} \frac{T_u - t_0 u}{A_{c,H}u^H} = x \in (-\infty, \infty]$, then

$$\psi_{\gamma,T_u}(u) = \mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}\psi_{0,T_u}(u)(1+o(1)), \quad u\to\infty. \tag{11}$$

Remarks. *a) For the case that $\gamma=1$ we can add: Under the statement i) above similar arguments as in the proof of Theorem 2.1 show that* (10) *holds as $u\to\infty$, with $\mathcal{M}_{H,1} = \mathcal{D}_H$ (see* (7)*). For ii) in Theorem 2.1, depending on the values of x different asymptotics will appear; those derivations are more involved and will therefore be omitted here.*
b) As discussed in [4, 25] also of interest is the investigation of the maximum losses given that ruin occurs, which, in our setup, is defined as

$$L(\gamma,u) := \left(\sup_{t\in[0,T_u]} W_\gamma(t) - u\right)\Bigg|(\tau_{\gamma,u}\le T_u). \tag{12}$$

Under the assumptions of Theorem 2.1, we have that if i) is satisfied, then

$$(1+cs_0)\frac{uL(\gamma,u)}{T_u^{2H}} \stackrel{d}{\to} \mathcal{E}, \quad u\to\infty,$$

and if ii) is valid, then

$$\frac{c^{2H}(1-H)^{2H-1}}{H^{2H}}\frac{L(\gamma,u)}{u^{2H-1}} \stackrel{d}{\to} \mathcal{E}, \quad u\to\infty.$$

Here (and in the sequel) $\mathcal{E}$ denotes a unit exponential random variable. Note in passing that the last convergence in distribution is clear when $\gamma=0$, $H=1/2$ and $T_u=\infty$ since it is known that the random variable $\sup_{t\in[0,\infty)} W_0(t)$ is exponentially distributed with parameter $2c$.

Next, we establish approximations for the conditional ruin times. It turns out that for the long time horizon the (scaled) conditional ruin time can be approximated by a truncated Gaussian random variable. Surprisingly, this is no longer the case for the short and the intermediate time horizons where the (scaled) conditional ruin time is approximated by an exponential random variable.

Theorem 2.2 *Let W_γ be the γ-reflected process defined in (1) with $H\in(0,1)$ and $\gamma\in(0,1)$, and let $\tau_{\gamma,u}$ be the ruin time defined as in (2). We have*

1) If $\lim_{u\to\infty} T_u/u = 0$, then

$$\frac{Hu^2(T_u-\tau_{\gamma,u})}{T_u^{2H+1}}\Big|(\tau_{\gamma,u}\le T_u) \stackrel{d}{\to} \mathcal{E}, \quad u\to\infty.$$

2) If $\lim_{u\to\infty} T_u/u = s_0\in(0,t_0)$, then

$$\frac{(1+cs_0)(H-(1-H)cs_0)(T_u-\tau_{\gamma,u})}{s_0^{2H+1}u^{2H-1}}\Big|(\tau_{\gamma,u}\le T_u) \stackrel{d}{\to} \mathcal{E}, \quad u\to\infty.$$

3) If $\lim_{u\to\infty} \frac{T_u-t_0u}{A_{c,H}u^H} = x\in(-\infty,\infty]$, then

$$\frac{\tau_{\gamma,u}-t_0u}{A_{c,H}u^H}\Big|(\tau_{\gamma,u}\le T_u) \stackrel{d}{\to} \mathcal{N}\Big|(\mathcal{N}<x), \quad u\to\infty.$$

3 Proofs

In this section, we shall present the proofs of both theorems displayed in Section 2. We start with the proof of Theorem 2.1. First note that for any $u>0$

$$\begin{aligned}\psi_{\gamma,T_u}(u) &= \mathbb{P}\left(\sup_{t\in[0,T_u]} W_\gamma(t)>u\right)\\ &= \mathbb{P}\left(\sup_{0\le s\le t\le T_u}\Big(Z(s,t)-c(t-\gamma s)\Big)>u\right),\end{aligned}$$

where $Z(s,t) := X_H(t) - \gamma X_H(s), s,t \ge 0$. Further, by the self-similarity of the fBm X_H

$$\psi_{\gamma,T_u}(u) = \mathbb{P}\left(\sup_{0\le s\le t\le 1} Y_u(s,t) > \frac{u}{T_u^H}\right), \tag{13}$$

where, for any $u > 0$

$$Y_u(s,t) = \frac{Z(s,t)}{1 + \frac{cT_u}{u}(t-\gamma s)}, \quad s,t \ge 0. \tag{14}$$

For the proof of statement $i)$ in Theorem 2.1 we shall make use of the following result.

Lemma 3.1 *Let $\{Y_u(s,t), s,t \ge 0\}, u > 0$ be a family of Gaussian random fields defined as in (14) with $H \in (0,1)$ and $\gamma \in (0,1)$. If the condition of statement $i)$ in Theorem 2.1 is satisfied, then for any u large enough, the variance function $V_{Y_u}^2(s,t) = \mathbb{E}\left(Y_u^2(s,t)\right)$ of the Gaussian random field Y_u attains its maximum over the set $\boldsymbol{E} := \{(s,t) : 0 \le s \le t \le 1\}$ at the unique point $(0,1)$. Moreover,*

$$V_{Y_u}(0,1) = \frac{u}{u + cT_u}$$

holds for all $u > 0$.

Proof of Lemma 3.1 First note that direct calculations yield

$$V_{Y_u}^2(s,t) = \frac{D(s,t)}{(1 + \frac{cT_u}{u}(t-\gamma s))^2}$$

with $D(s,t) = (1-\gamma)t^{2H} + (\gamma^2-\gamma)s^{2H} + \gamma(t-s)^{2H}$. It follows further that

$$\begin{aligned}
\frac{\partial V_{Y_u}^2(s,t)}{\partial s} &= \left(1 + \frac{cT_u}{u}(t-\gamma s)\right)^{-4} \left((2H(\gamma^2-\gamma)s^{2H-1} - 2H\gamma(t-s)^{2H-1}) \left(1 + \frac{cT_u}{u}(t-\gamma s)\right)^2 \right. \\
&\quad \left. + 2\gamma \frac{cT_u}{u} D(s,t) \left(1 + \frac{cT_u}{u}(t-\gamma s)\right)\right), \\
\frac{\partial V_{Y_u}^2(s,t)}{\partial t} &= \left(1 + \frac{cT_u}{u}(t-\gamma s)\right)^{-4} \left((2H(1-\gamma)t^{2H-1} + 2H\gamma(t-s)^{2H-1}) \left(1 + \frac{cT_u}{u}(t-\gamma s)\right)^2 \right. \\
&\quad \left. - 2\frac{cT_u}{u} D(s,t) \left(1 + \frac{cT_u}{u}(t-\gamma s)\right)\right).
\end{aligned}$$

Thus if $\frac{\partial V_{Y_u}^2(s,t)}{\partial s} = \frac{\partial V_{Y_u}^2(s,t)}{\partial t} = 0$, then

$$s^{2H-1} + (t-s)^{2H-1} = t^{2H-1}. \tag{15}$$

Moreover, since $2H - 1 < 1$ the above does not hold in the interior of the set $\boldsymbol{E}$. Therefore, we conclude that the maximum point of $V_{Y_u}^2(s,t)$ over $\boldsymbol{E}$ is on one of the three lines $l_1 = \{(0,t), 0 \le t \le 1\}$, $l_2 = \{(s,t), 0 \le s = t \le 1\}$ or $l_3 = \{(s,1), 0 \le s \le 1\}$. It can be shown that on l_1 the maximum is attained uniquely at $(0,1)$ and on l_2 the maximum is attained uniquely at $(1,1)$. Clearly, both $(0,1)$ and $(1,1)$ lie on the line l_3. Consequently, the maximum point of $V_{Y_u}^2(s,t)$ over $\boldsymbol{E}$ is on l_3. Moreover, we have that

$$\left(V_{Y_u}^2(s,1)\right)' = \frac{2c\gamma T_u}{u}\left(1+\frac{cT_u}{u}(1-\gamma s)\right)^{-3} f_{\frac{cT_u}{u}}(s),$$

where, for any $d>0$

$$\begin{aligned} f_d(s) &= 1-\gamma-(\gamma-\gamma^2)s^{2H}+\gamma(1-s)^{2H}-\frac{H}{d}(1+d-d\gamma s) \\ &\quad \times\left((1-\gamma)s^{2H-1}+(1-s)^{2H-1}\right), \quad s\geq 0. \end{aligned}$$

Next, we show that for any $\gamma\in[0,1)$ and $d\in[0,\frac{H}{1-H})$

$$f_d(s)<0, \quad \forall s\in(0,1) \tag{16}$$

holds. Let us first rewrite $f_d(s)$ as

$$\begin{aligned} f_d(s) &= (1-\gamma)+\gamma(1-H)(1-s)^{2H}-\gamma(1-\gamma)(1-H)s^{2H} \\ &\quad -H\left(1+\frac{1}{d}-\gamma\right)(1-s)^{2H-1}-\frac{H}{d}(1+d)(1-\gamma)s^{2H-1}. \end{aligned}$$

Note that

$$1-\gamma<(1-\gamma)(1-s)^{2H-1}+(1-\gamma)s^{2H-1}, \quad s\in(0,1),$$

hence replacing $1-\gamma$ by $(1-\gamma)(1-s)^{2H-1}+(1-\gamma)s^{2H-1}$ in the above equation yields

$$\begin{aligned} f_d(s) &< (1-\gamma)(1-s)^{2H-1}+(1-\gamma)s^{2H-1}+\gamma(1-H)(1-s)^{2H}-\gamma(1-\gamma)(1-H)s^{2H} \\ &\quad -H\left(1+\frac{1}{d}-\gamma\right)(1-s)^{2H-1}-\frac{H}{d}(1+d)(1-\gamma)s^{2H-1} \\ &< \left(1-H-\frac{H}{d}\right)\left((1-s)^{2H-1}+(1-\gamma)s^{2H-1}\right)-\gamma(1-\gamma)(1-H)s^{2H}, \end{aligned}$$

where in the second inequality we used the fact that

$$\gamma(1-H)(1-s)^{2H}\leq\gamma(1-H)(1-s)^{2H-1}, \quad \forall s\in(0,1).$$

Since for any $d\in[0,\frac{H}{1-H})$

$$1-H<\frac{H}{d}$$

we conclude that (16) is valid. Consequently, by (16) and the fact that

$$\lim_{u\to\infty}\frac{cT_u}{u}=cs_0<\frac{H}{1-H}$$

we obtain

$$\left(V_{Y_u}^2(s,1)\right)' < 0, \qquad \forall s \in (0,1).$$

Hence the maximum of $V_{Y_u}^2(s,t)$ over the set $\boldsymbol{E}$ is attained at the unique point $(0,1)$. This completes the proof. $\Box$

Proof of Theorem 2.1 *i*). First, note that (13) can be rewritten as

$$\psi_{\gamma,T_u}(u) = \mathbb{P}\left(\sup_{0\le s\le t\le 1} \frac{Y_u(s,t)}{V_{Y_u}(0,1)} > \frac{u+cT_u}{T_u^H}\right).$$

Next, in order to establish the claim by applying Theorem 4.1 (see Appendix) we need to verify the assumptions **A1–A3** therein. This can be done by employing similar arguments as in the proof of Theorem 2.5 in [20]. Indeed, it follows that for any fixed large u

$$\frac{V_{Y_u}(s,t)}{V_{Y_u}(0,1)} = \begin{cases} 1-(H-c(u))(1-t)-\gamma(H-c(u))s+o(1-t+s), & H>1/2,\\ 1-(\frac{1}{2}-c(u))(1-t)-\gamma(1-\frac{\gamma}{2}-c(u))s+o(1-t+s), & H=1/2,\\ 1-(H-c(u))(1-t)-\frac{\gamma-\gamma^2}{2}s^{2H}+o(1-t+s^{2H}), & H<1/2 \end{cases} \tag{17}$$

holds as $(s,t)\to(0,1)$, where $c(u)=\frac{cT_u}{u+cT_u}$. Furthermore, for any $u>0$

$$1-Cov\left(\frac{Y_u(s,t)}{V_{Y_u}(s,t)}, \frac{Y_u(s',t')}{V_{Y_u}(s',t')}\right) = \frac{1}{2}\left(\mid t-t'\mid^{2H} + \gamma^2 \mid s-s'\mid^{2H}\right)(1+o(1)) \tag{18}$$

holds as $(s,t),(s',t')\to(0,1)$. In addition, there exists a positive constant $\mathbb{Q}$ such that, for all u large enough

$$\mathbb{E}\left(\left(\frac{Y_u(s,t)}{V_{Y_u}(0,1)} - \frac{Y_u(s',t')}{V_{Y_u}(0,1)}\right)^2\right) \le \mathbb{Q}(|t-t'|^{2H}+|s-s'|^{2H})$$

holds for all $(s,t)\in\boldsymbol{E}$. Therefore, by the fact that

$$\lim_{u\to\infty} c(u) = c_0 = \frac{cs_0}{1+cs_0} < H$$

and using Theorem 4.1 we obtain that

$$\psi_{\gamma,T_u}(u) = D_{H,\gamma}\left(\frac{u+cT_u}{T_u^H}\right)^{(\frac{1-2H}{H})_+} \Psi\left(\frac{u+cT_u}{T_u^H}\right)(1+o(1)) \tag{19}$$

as $u\to\infty$, where

$$\mathcal{D}_{H,\gamma} = \begin{cases} 2^{-\frac{1}{2H}}(H-c_0)^{-1}\mathcal{H}_{2H}\mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}, & \text{if } H<1/2,\\ \frac{4(1-c_0)^2}{(1-2c_0)(2-2c_0-\gamma)}, & \text{if } H=1/2,\\ 1 & \text{if } H>1/2. \end{cases}$$

Combining the above formula with (6) we obtain (10).

Next, we present the proof of statement ii). Assume first that

$$\lim_{u\to\infty} \frac{T_u - t_0 u}{A_{c,H} u^H} = x \in \mathbb{R}.$$

In view of (4)

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{\tau_{\gamma,u} - t_0 u}{A_{c,H} u^H} \le x \Big| \tau_{\gamma,u} < \infty \right) \;=\; \Phi(x)$$

Clearly, the above is equivalent to

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(\sup_{0\le t\le t_0 u + x A_{c,H} u^H} W_\gamma(t) > u\right)}{\mathbb{P}\left(\tau_{\gamma,u} < \infty\right)} = \Phi(x).$$

implying

$$\psi_{\gamma,T_u}(u) \;=\; \psi_{\gamma,\infty}(u)\Phi(x)(1+o(1))$$

as $u \to \infty$, which together with (8) and Theorem 1.1 in [20] yields the validity of (11). Finally, assume that $\lim_{u\to\infty} \frac{T_u - t_0 u}{A_{c,H} u^H} = \infty$. For any positive large M

$$\psi_{\gamma, t_0 u + M A_{c,H} u^H}(u) \le \psi_{\gamma,T_u}(u) \le \psi_{\gamma,\infty}(u)$$

holds for all u large enough, hence

$$\Phi(M) \le \liminf_{u\to\infty} \frac{\psi_{\gamma,T_u}(u)}{\psi_{\gamma,\infty}(u)} \le \limsup_{u\to\infty} \frac{\psi_{\gamma,T_u}(u)}{\psi_{\gamma,\infty}(u)} \le 1.$$

Letting thus $M \to \infty$ yields

$$\psi_{\gamma,T_u}(u) \;=\; \psi_{\gamma,\infty}(u)(1+o(1)), \quad u \to \infty.$$

Consequently, (11) is valid, and thus the claim follows. $\square$

Proof of Theorem 2.2 We start with the proof of statement 1). Set $T_x(u) = T_u - x T_u^{2H+1}/u^2$. It follows from (19) that, for any $x > 0$

$$\begin{aligned}
\mathbb{P}\left(\frac{u^2(T_u - \tau_{\gamma,u})}{T_u^{2H+1}} > x \Big| \tau_{\gamma,u} \le T_u \right) &= \frac{\mathbb{P}\left(\sup_{0\le t\le T_x(u)} W_\gamma(t) > u\right)}{\mathbb{P}\left(\sup_{0\le t\le T_u} W_\gamma(t) > u\right)} \\
&= \frac{D_{H,\gamma}\left(\frac{u+cT_x(u)}{(T_x(u))^H}\right)^{(\frac{1-2H}{H})_+} \Psi\left(\frac{u+cT_x(u)}{(T_x(u))^H}\right)}{D_{H,\gamma}\left(\frac{u+cT_u}{T_u^H}\right)^{(\frac{1-2H}{H})_+} \Psi\left(\frac{u+cT_u}{T_u^H}\right)}(1+o(1)) \\
&= \exp\left(-\frac{\left(\frac{u+cT_x(u)}{(T_x(u))^H}\right)^2 - \left(\frac{u+cT_u}{T_u^H}\right)^2}{2} \right)(1+o(1))
\end{aligned}$$

$$\to \quad \exp(-Hx)$$

holds as $u \to \infty$ establishing the claim.

Next, we give the proof of statement 2). Similar arguments as above yield that, for any $x > 0$

$$\begin{aligned}\mathbb{P}\left(\frac{T_u - \tau_{\gamma,u}}{u^{2H-1}} > x \Big| \tau_{\gamma,u} \le T_u\right) &= \exp\left(-\frac{\left(\frac{u+c(T_u - xu^{2H-1})}{(T_u - xu^{2H-1})^H}\right)^2 - \left(\frac{u+cT_u}{T_u^H}\right)^2}{2}\right)(1+o(1)) \\ &\to \exp(-\lambda x), \quad u \to \infty,\end{aligned}$$

where $\lambda = \frac{(1+cs_0)(H-(1-H)cs_0)}{s_0^{2H+1}}$. Finally, since by (11) for any $y \le x$

$$\begin{aligned}\mathbb{P}\left(\frac{\tau_{\gamma,u} - t_0 u}{A_{c,H}u^H} < y \Big| \tau_{\gamma,u} \le T_u\right) &= \frac{\mathbb{P}\left(\sup_{0 \le t \le t_0 u + yA_{c,H}u^H} W_\gamma(t) > u\right)}{\mathbb{P}\left(\sup_{0 \le t \le T_u} W_\gamma(t) > u\right)} \\ &\to \frac{\Phi(y)}{\Phi(x)}, \quad u \to \infty\end{aligned}$$

the claim of statement 3) follows, and thus the proof is complete. $\square$

4 Appendix

We present below a generalization of Theorem D.3 and Theorem 8.2 in [26], which is tailored for the proof of our main results. We first introduce the Piterbarg constant $\widetilde{\mathcal{P}}_\alpha^b, \alpha \in (0,2), b > 0$ defined by

$$\widetilde{\mathcal{P}}_\alpha^b = \lim_{S \to \infty} \mathbb{E}\left(\exp\left(\sup_{t \in [-S,S]}\left(\sqrt{2}X_{\alpha/2}(t) - (1+b)|t|^\alpha\right)\right)\right) \in (0,\infty),$$

where $X_{\alpha/2}$ is a standard fBm defined on $\mathbb{R}$; see also Theorem 2.1 for the Piterbarg constant $\mathcal{P}_\alpha^b$. Set $\boldsymbol{E} = \{(s,t), 0 \le s \le t \le 1\}$ and let $\{\eta_u(s,t), (s,t) \in \boldsymbol{E}\}, u \ge 0$ be a family of Gaussian random fields satisfying the following three assumptions:

A1: The variance function $\sigma_{\eta_u}^2(s,t)$ of η_u attains its muximum on the set $\boldsymbol{E}$ at some unique point (s_0,t_0) for any u large enough, and further there exist four positive constants $A_i, \beta_i, i = 1,2$ and two functions $A_i(u), i = 1,2$ satisfying $\lim_{u\to\infty} A_i(u) = A_i, i = 1,2$ such that $\sigma_{\eta_u}(s,t)$ has the following expansion for u large

$$\sigma_{\eta_u}(s,t) = 1 - A_1(u)|s-s_0|^{\beta_1}(1+o(1)) - A_2(u)|t-t_0|^{\beta_2}(1+o(1)), \quad (s,t) \to (s_0,t_0).$$

A2: There exist four constants $B_i > 0, \alpha_i \in (0,2), i = 1,2$ and two functions $B_i(u), i = 1,2$ satisfying $\lim_{u\to\infty} B_i(u) = B_i, i = 1,2$ such that the correlation function $r_{\eta_u}(s,t;s',t')$ of η_u has the following expansion for all u large

$$r_{\eta_u}(s,t;s',t') = 1 - B_1(u)|s-s'|^{\alpha_1}(1+o(1)) - B_2(u)|s-s'|^{\alpha_2}(1+o(1)), \quad (s,t),(s',t') \to (s_0,t_0).$$

A3: For some positive constants $\mathbb{Q}$ and γ, and all u large enough

$$\mathbb{E}\left(\eta_u(s,t)-\eta_u(s',t')\right)^2 \le \mathbb{Q}(|s-s'|^\gamma+|t-t'|^\gamma)$$

holds for any $(s,t),(s',t')\in \boldsymbol{E}$.

Theorem 4.1 *If $\{\eta_u(s,t),(s,t)\in \boldsymbol{E}\}$, $u\ge 0$ is a family of Gaussian random fields satisfying* **A1–A3**, *then*

$$\mathbb{P}\left(\sup_{(s,t)\in D}\eta_u(s,t)>u\right)=\mathcal{F}^{(1)}_{\alpha,\beta}(u)\ \mathcal{F}^{(2)}_{\alpha,\beta}(u)\ \Psi(u),\quad \textit{as } u\to\infty,$$

where

$$\mathcal{F}^{(i)}_{\alpha,\beta}(u)=\begin{cases}\widehat{I}_i\mathcal{H}_{\alpha_i}B_i^{\frac{1}{\alpha_i}}A_i^{-\frac{1}{\beta_i}}\Gamma\left(\frac{1}{\beta_i}+1\right)\ u^{\frac{2}{\alpha_i}-\frac{2}{\beta_i}}, & \textit{if } \alpha_i<\beta_i,\\ \widehat{\mathcal{P}}_{\alpha_1}^{\frac{A_i}{B_i}}, & \textit{if } \alpha_i=\beta_i,\\ 1 & \textit{if } \alpha_i>\beta_i,\end{cases}\qquad i=1,2,$$

with $\Gamma(\cdot)$ the Euler Gamma function and

$$\widehat{\mathcal{P}}_{\alpha_1}^{\frac{A_1}{B_1}}=\begin{cases}\widetilde{\mathcal{P}}_{\alpha_1}^{\frac{A_1}{B_1}}, & \textit{if } s_0\in(0,1),\\ \mathcal{P}_{\alpha_1}^{\frac{A_1}{B_1}} & \textit{if } s_0=0 \textit{ or } 1,\end{cases}\qquad \widehat{\mathcal{P}}_{\alpha_2}^{\frac{A_2}{B_2}}=\begin{cases}\widetilde{\mathcal{P}}_{\alpha_2}^{\frac{A_2}{B_2}}, & \textit{if } t_0\in(0,1),\\ \mathcal{P}_{\alpha_1}^{\frac{A_2}{B_2}} & \textit{if } t_0=0 \textit{ or } 1,\end{cases}$$

$$\widehat{I}_1=\begin{cases}2, & \textit{if } s_0\in(0,1),\\ 1 & \textit{if } s_0=0 \textit{ or } 1,\end{cases}\qquad \widehat{I}_2=\begin{cases}2, & \textit{if } t_0\in(0,1),\\ 1 & \textit{if } t_0=0 \textit{ or } 1.\end{cases}$$

Proof of Theorem 4.1 It follows from the assumptions **A1–A2** that for any $\varepsilon>0$ and for u large enough we have

$$(A_1-\varepsilon)|s-s_0|^{\beta_1}+(A_2-\varepsilon)|t-t_0|^{\beta_2}\le 1-\sigma_{\eta_u}(s,t)\le (A_1+\varepsilon)|s-s_0|^{\beta_1}+(A_2+\varepsilon)|t-t_0|^{\beta_2}$$

as $(s,t)\to(s_0,t_0)$, and

$$(B_1-\varepsilon)|s-s'|^{\alpha_1}+(B_2-\varepsilon)|t-t'|^{\alpha_2}\le 1-r_{\eta_u}(s,t;s',t')\le (B_1+\varepsilon)|s-s'|^{\alpha_1}+(B_2+\varepsilon)|t-t'|^{\alpha_2}$$

as $(s,t),(s',t')\to(s_0,t_0)$. Therefore, in the light of Theorem 8.2 in [26] we can get appropriate asymptotical upper and lower bounds, and thus the claims follow by letting $\varepsilon\to 0$. The proof is complete. $\square$

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